

The motion of a charged particle in Kaluza-Klein manifolds

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Abstract

In this paper we use Jacobi fields to describe the motion of a charged particle in the classical gravitational, electromagnetic, and Yang-Mills fields.

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1 Introduction

The Jacobi fields are very important to the Riemannian Geometry[1] and in singularity theorems[2]. These fields can also be used to study the motion of a particle in a Riemannian spacetime as an alternative to the usual geodesic approach. The general relativity theory assures us that every coordinate frame is physically equivalent from a classical point of view. Thus we can claim that any falling particle in a gravitational field will be accelerated relatively to any stationary coordinate frame defined by a Killing vector. However, it is possible that a specific spacetime does not have Killing vector, so in this case we would not have at our disposal a frame which enables us to measure this acceleration. In such a situation, the best we can do is to take two different particles and measure their relative acceleration. This is a relativistic three-body problem. The metric deformation by test particles[3] is necessary in a more realistic approach to the Kaluza-Klein manifolds, but it is a difficult problem in geometry so that we will not consider it. In this case a relative acceleration will be given by the equation which governs the Jacobi fields.

This paper is organized as follows. In Sec.2 we present some facts about the Jacobi fields. In Sec.3 we build a Kaluza-Klein geometry and the associated Jacobi equation. In Sec.4 we summarize the main results of this work.

2 Jacobi Fields

In this section we briefly review the Jacobi fields and their respective differential equation for a Riemann manifold. Let us consider a differentiable manifold, \mathcal{M} , and two structures defined on \mathcal{M} , namely affine connection, ∇ , and Riemann tensor, K , related by the equation[2]

$$K(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z, \quad (2.1)$$

where X, Y and Z are vector fields in the tangent space. The torsion tensor can be defined by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \quad (2.2)$$

where $\nabla_X Y$ is the covariant derivative of the field Y along the X direction. We define the Lie derivative by

$$\mathcal{L}_X Y = [X, Y]. \quad (2.3)$$

For a torsion-free connection we can write

$$\mathcal{L}_X Y = \nabla_X Y - \nabla_Y X. \quad (2.4)$$

Let us consider the tangent field V , the Jacobi fields Z and the condition

$$\mathcal{L}_V Z = 0. \quad (2.5)$$

From (2.5), we can easily see that the equations which govern the Jacobi field can be written in the form

$$\nabla_V \nabla_V Z + K(Z, V)V - \nabla_Z \nabla_V V = 0. \quad (2.6)$$

We assume that

$$\nabla_V V = 0. \quad (2.7)$$

In this case the Jacobi equations are reduced to a geodesic deviation, assuming a simpler form, and the Fermi derivative $\frac{D_F Z}{\partial s}$ coincides with the usual covariant derivative

$$\frac{D_F Z}{\partial s} = \frac{DZ}{\partial s} = \nabla_V Z,$$

and

$$\nabla_V \nabla_V Z + K(Z, V)V = 0. \quad (2.8)$$

3 Jacobi equation in a Kaluza-Klein manifold

We intend to make use of a vielbein basis, but it is necessary, beforehand, to present a local Kaluza-Klein metric G for the fields[4, 5, 6]. The metric tensor G has signature $(-, +, \dots, +)$. For Kaluza-Klein manifold the indices $\Lambda, \Pi \in (0, 1, 2, 3, 4, 5, 6, \dots, n)$ and for ordinary space-time the indices

$\alpha, \beta \in (0, 1, 2, 3)$. For internal space we have indices $x^4 = y$ for the electromagnetic fields and $i, j \in (5, 6, \dots, n)$ for the Yang-Mills fields. We can write a local basis coordinate system as $x^\Lambda = (x^\alpha, y, z^i)$. In this basis the metric G is given by the ansatz

$$G_{\Lambda\Pi} = g_{\alpha\beta} + \eta_{(4)(4)} A_\alpha A_\beta + h_{ij} \xi_m^i \xi_n^j A_\alpha^m A_\beta^n, \quad (3.1)$$

$$G_{\alpha 4} = \eta_{(4)(4)} A_\alpha, \quad (3.2)$$

$$G_{i4} = 0, \quad (3.3)$$

$$G_{\alpha i} = h_{ij} \xi_m^i A_\alpha^m, \quad (3.4)$$

where the form invariance of $h_{ij}(z)$ was assumed and this implies that Killing equation is obeyed, or equivalently

$$L_\xi h = 0. \quad (3.5)$$

Next, let us consider the connection between the vielbein and the local metric tensor

$$G_{\Lambda\Pi} = E_\Lambda^{(\mathbf{A})} E_\Pi^{(\mathbf{B})} \eta_{(\mathbf{A})(\mathbf{B})}, \quad (3.6)$$

where $\eta_{(\mathbf{A})(\mathbf{B})}$ and $E_\Lambda^{(\mathbf{A})}$ are Lorentzian metric and vielbein components respectively. The flat indices $(\mathbf{A}), (\mathbf{B}), \dots, (\mathbf{M}), (\mathbf{N}) \in (0, 1, 2, 3, 4, 5, 6, \dots, n)$. Specifically we have $(\mathbf{A}) = (A, (4), a)$, with $A \in (0, 1, 2, 3)$, $a \in (5, 6, \dots, n)$, so that the Lorentzian metric is composed as follows

$$\eta_{(\mathbf{A})(\mathbf{B})} = (\eta_{AB}, \eta_{(4)(4)}, \eta_{ab}), \quad (3.7)$$

where η_{AB} are the flat metric of the ordinary space-time and $\eta_{(4)(4)}, \eta_{ab}$ of the internal space, are associated with the electromagnetic and Yang-Mills fields respectively. In the vielbein basis we use the following Riemannian curvature

$$\begin{aligned} K_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})} &= -\gamma_{(\mathbf{A})(\mathbf{B})(\mathbf{C}),(\mathbf{D})} + \gamma_{(\mathbf{A})(\mathbf{B})(\mathbf{D}),(\mathbf{C})} \\ &+ \eta^{(\mathbf{M})(\mathbf{N})} [\gamma_{(\mathbf{B})(\mathbf{A})(\mathbf{M})} (\gamma_{(\mathbf{C})(\mathbf{N})(\mathbf{D})} - \gamma_{(\mathbf{D})(\mathbf{N})(\mathbf{C})}) \\ &+ \gamma_{(\mathbf{M})(\mathbf{A})(\mathbf{C})} \gamma_{(\mathbf{B})(\mathbf{N})(\mathbf{D})} - \gamma_{(\mathbf{M})(\mathbf{A})(\mathbf{D})} \gamma_{(\mathbf{B})(\mathbf{N})(\mathbf{C})}], \end{aligned} \quad (3.8)$$

where the $\gamma_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})}$ are the Ricci rotation coefficients. The only non-vanishing coefficients are

$$\gamma_{aBC} = 1/2 \xi_{ma} F_{AB}^m, \quad (3.9)$$

$$\gamma_{ABc} = -1/2\xi_{mc}F_{AB}^m, \quad (3.10)$$

$$\gamma_{(4)AB} = 1/2\eta_{(4)(4)}F_{AB}, \quad (3.11)$$

$$\gamma_{AB(4)} = -1/2\eta_{(4)(4)}F_{AB}, \quad (3.12)$$

$$\gamma_{ABC} = E_A^\Lambda E_{\Lambda B;\Pi} E_C^\Pi, \quad (3.13)$$

with

$$F_{AB} = A_{B,A} - A_{A,B}, \quad (3.14)$$

and

$$F_{AB}^m = A_{B,A}^m - A_{A,B}^m + f_{ln}^m A_A^l A_B^n, \quad (3.15)$$

where $A_A = A_A(x)$, $A_A^m = A_A^m(x)$, $\xi_{ma} = \xi_{ma}(z)$, and $\xi_{ma} = \eta_{mb}\xi_a^b$. Using the above results in (3.8) we obtain the following Riemannian components

$$\begin{aligned} K_{ABCD} = & R_{ABCD} + 1/4\eta_{(4)(4)}[-2F_{AB}F_{CD} - F_{AC}F_{BD} \\ & + F_{AD}F_{BC}] + 1/4\eta^{ab}\xi_{ma}\xi_{lb}[-2F_{AB}^mF_{CD}^l \\ & - F_{AC}^mF_{BD}^l + F_{AD}^mF_{BC}^l], \end{aligned} \quad (3.16)$$

$$\begin{aligned} K_{abCD} = & \gamma_{abd}\xi_m^d F_{CD}^m \\ & - 1/4[\xi_{la}\xi_{mb} - \xi_{lb}\xi_{ma}]\eta^{MN}F_{NC}^l F_{MD}^m, \end{aligned} \quad (3.17)$$

$$K_{(4)B(4)C} = 1/4\eta_{(4)(4)}^2\eta^{MN}F_{MB}F_{NC}, \quad (3.18)$$

$$\begin{aligned} K_{(4)BCD} = & 1/2\eta_{(4)(4)}(F_{BD,C} - F_{BC,D}) \\ & + \eta^{MN}[(\gamma_{DNC} - \gamma_{CND})F_{BM} - \gamma_{BND}F_{MC} + \gamma_{BNC}F_{MD}], \end{aligned} \quad (3.19)$$

$$\begin{aligned} K_{aBCD} = & 1/2\xi_{ma}\{F_{BC,D}^m - F_{BD,C}^m + \eta^{MN}[(\gamma_{DNC} \\ & - \gamma_{CND})F_{BM}^m + \gamma_{BNC}F_{MD}^m - \gamma_{BND}F_{MC}^m]\}, \end{aligned} \quad (3.20)$$

where $R_{ABCD}(x)$ are the Riemannian components in the ordinary spacetime. It is important to note that the following properties can be verified

$$K_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})} = K_{(\mathbf{C})(\mathbf{D})(\mathbf{A})(\mathbf{B})}, \quad (3.21)$$

and

$$K_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})} + K_{(\mathbf{A})(\mathbf{D})(\mathbf{B})(\mathbf{C})} + K_{(\mathbf{A})(\mathbf{C})(\mathbf{D})(\mathbf{B})} = 0, \quad (3.22)$$

where for some set of indices we have used (3.5).

We now consider a massive test particle. Since the particle has a non-vanishing rest mass, it is convenient to define the tangent vector V as a timelike one, so that $g(V, V) = -1$. Let us build a Fermi-Walker transport. In the Fermi-Walker transported particle frame the equation (2.8) is given by

$$\frac{d^2 Z_{(\mathbf{A})}}{d\tau^2} + K_{0(\mathbf{A})0(\mathbf{C})} Z_{(\mathbf{C})} = 0. \quad (3.23)$$

More specifically:

$$\frac{d^2 Z_{\tilde{A}}}{d\tau^2} + K_{0\tilde{A}0\tilde{C}} Z_{\tilde{C}} + K_{0\tilde{A}0(4)} Z_{(4)} + K_{0\tilde{A}0a} Z_a = 0, \quad (3.24)$$

$$\frac{d^2 Z_{(4)}}{d\tau^2} + K_{0(4)0(4)} Z_{(4)} = 0, \quad (3.25)$$

and

$$\frac{d^2 Z_a}{d\tau^2} + K_{0a0b} Z_b + K_{0a0\tilde{C}} Z_{\tilde{C}} = 0, \quad (3.26)$$

where τ is, in general, an affine parameter, which in our case is the proper time of the particle and $Z_{(\mathbf{A})}$ are the vielbein components of the space-like vector Z , with $g(Z, V) = 0$ and $\tilde{A}, \tilde{C} = (1, 2, 3)$.

4 Concluding Remarks

The equations (3.24), (3.25), and (3.26) indicate that the particle motion is a function of the gravitational, electromagnetic and Yang-Mills fields. The equation (3.24) suggests that the particle motion in a ordinary spacetime can depend on matter constituent details, but this effect is very small when compared to the others. We do not expect that (3.24)-(3.26) are a representation of the complete ansatz. The metric deformation by test particles[3] is necessary, although it is not enough in a more realistic approach. Actually, it is an open problem in geometry. We believe that the results obtained in the present paper might lead to a better comprehension of these issues in the future. ¹

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